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A Functional Equation Method for Ordinary Differential Equations

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1. INTRODUCTION

The problem of obtaining an exact analytical expression $f(x)$ for the general solution of the real scalar nonlinear differential equation

$$\frac{df(x)}{dx} = F(x, f(x)) \tag{1}$$

is still of interest, especially when one or more arbitrary parameters with unbounded range occur in it, in which case numerical and approximate methods can be of only limited usefulness. This problem will be here formulated in a different manner as that of obtaining and solving a functional equation in a single variable satisfied by $f(x)$. Such a functional equation has, in general, a very large class of solutions, but Eq. (1) provides the restriction on this class which is needed in order to obtain the required solution. The theory of functional equations in a single variable has recently been extensively developed and many of the results and methods of solution of these equations are given in the book by Kuczma [1].

Our method of obtaining a functional equation for $f(x)$ (as defined by (1)) consists of assuming a simple form for the functional equation, containing three unknown functions. These functions satisfy a single second-order differential equation, and we must determine any particular set which fulfills the equation. Although the differential equation is of the second order, the facts that it is only one equation for three dependent variables and that only a particular solution set is required, may make it easier to handle than the original equation. For instance, it may be possible to choose one or two of the functions so as to simplify the resulting differential equation. Having determined these three functions, the functional equation may now be attacked by the available methods, subject to the restriction provided by Eq. (1).

The method of functional equations may, in principle, be extended to differential equations of higher order and to systems of equations; for instance, for a second-order equation one proceeds just as for the first-order case and obtains a single fourth-order differential equation for the three unknown functions. Of course, there is no guarantee that one can determine these functions, even for the first-order equation, but the method is worth a try when other methods have failed, and here manipulative skill is decisive for its success.

As an example for the application of our method we shall consider a Riccati equation containing an arbitrary parameter, find its functional equation and solve it to obtain the general solution $f(x)$. More precisely, two functional equations are inadvertently obtained, one of them being valid for only a particular value of the parameter; this value separates two types of the solution curves: non-oscillatory and oscillatory. In contrast, it does not seem possible to readily estimate this value of the parameter by manipulating the differential equation. This instance shows that the mere formulation of the functional equation may (at least) indicate a significant aspect of the solution of the differential equation.

2. The Functional Equation Method

Let us assume that the general solution $f(x)$ of (1) satisfies a linear functional equation of the form

$$f(x) = a(x) f(z(x)) + b(x), \tag{2}$$

where $a(x)$, $b(x)$ and $z(x)$ are functions to be determined.

Evidently, one can also assume a more general form such as

$$G(f(x)) = a(x) f(z(x)) + b(x),$$

but the general linear equation is here chosen because it seems to be the most widely studied functional equation in a single variable; special cases of it are the Schröder and the Abel equations, and a large part of [1] is devoted to the general linear equation and its special cases.

For the determination of the functions $a(x)$, $b(x)$ and $z(x)$ in (2), we shall obtain a single differential equation in three dependent variables. Differentiating (2) and making use of (1) and (2), we get

$$F(x, f(x)) = \frac{a'}{a}\{f(x) - b\} + az'F\left(z, \frac{f(x) - b}{a}\right) + b'. \tag{3}$$

Setting

$$\phi \equiv f(z) = \frac{f(x) - b(x)}{a(x)}$$

and differentiating (3), again making use of (1), we obtain

$$\begin{aligned}
&\frac{\partial F(x, f(x))}{\partial x} + F(x, f(x)) \frac{\partial F(x, f(x))}{\partial f(x)} \\
&\quad = \left(\frac{a'}{a}\right)' \{f(x) - b\} + \frac{a'}{a} \{F(x, f(x)) - b'\} \\
&\quad + (az')' F(z, \phi) + az'^2 \frac{\partial F(z, \phi)}{\partial z} \\
&\quad + z' \frac{\partial F(z, \phi)}{\partial \phi} \left\{F(x, f(x)) - b' - \frac{a'}{a} f(x) + \frac{a'b}{a}\right\} + b''. \qquad (4)
\end{aligned}$$

Assuming now that $f(x)$ can be eliminated between (3) and (4), we get for the eliminant an equation of the form

$$\psi(x; a, a', a''; b, b', b''; z, z', z'') = 0, \qquad (5)$$

which is a second-order differential equation for three dependent variables, any particular solution set of it being all that is needed in order to obtain the functional Eq. (2) for $f(x)$. We notice that any number of distinct functional equations of the form (2) can be obtained, corresponding to any particular triple $a(x)$, $b(x)$, $z(x)$ satisfying (5). This aspect illustrates the extreme generality of the solutions of functional equations.

We shall now illustrate the method by an example, and even though the general solution of the differential equation considered is already known, the example has an interesting feature which we wish to exhibit.

3. Application to a Riccati Equation

We consider the normal form of the Riccati equation

$$\frac{df(x)}{dx} = p(x) + f^2(x), \qquad (6)$$

for which Eq. (3) is a quadratic and (4) a cubic in $f(x)$, so $f(x)$ can be readily eliminated between them. However, if we set $a(x) \equiv z'(x)$, these reduce to a linear and a quadratic equation, respectively, and we get for the eliminant Eq. (A.1) of the Appendix, which is a differential equation for $b(x)$ and $z(x)$.

For a specific example, we consider

$$\frac{df(x)}{dx} = \frac{\alpha}{x^2} + f^2(x), \tag{7}$$

where α is a real parameter taking values in the interval $(-\infty, \infty)$. Setting $b = \beta x^r$ and $z = \gamma x^m$ in Eq. (A.1), we find that the equation is satisfied for $r = -1$ and either (i) $\alpha = \frac{1}{4}$, $\beta = (m-1)/2$ and m, γ arbitrary non-zero numbers, or (ii) $m = -1$, $\beta = -1$ and α, γ arbitrary, except that $\gamma \neq 0$. This result indicates that the value $\alpha = \frac{1}{4}$ is somehow significant for the character of the solution, as will appear below.

Thus, for $\alpha = \frac{1}{4}$, we get the functional equation

$$f(x) = m\gamma x^{m-1} f(\gamma x^m) + \frac{m-1}{2x}, \tag{8}$$

where m, γ are arbitrary non-zero numbers; and for all values of α (including $\alpha = \frac{1}{4}$), we obtain the functional equation

$$f(x) = -\frac{\gamma}{x^2} f\left(\frac{\gamma}{x}\right) - \frac{1}{x}, \tag{9}$$

where γ is an arbitrary non-zero number.

Now, to obtain a solution of (8) which is independent of γ, we differentiate (8) partially with respect to γ, using (7) with $\alpha = \frac{1}{4}$, to get

$$f(\gamma x^m) = -\frac{1}{2\gamma x^m}, \qquad \text{or} \qquad f(x) = -\frac{1}{2x},$$

which is a particular integral of (7).

Next, differentiating (8) partially with respect to m, we get a quadratic for $f(\gamma x^m)$; replacing γx^m by x, its solutions are

$$f_1(x) = -\frac{1}{2x} \qquad \text{and} \qquad f_2(x) = \frac{1}{x(\log \gamma - \log x)} - \frac{1}{2x},$$

the latter being the general integral of (7) for $\alpha = \frac{1}{4}$, since γ is arbitrary.

Differentiating now (9) partially with respect to γ, we get

$$f(x) = \frac{k}{x}, \qquad \text{where} \qquad k^2 + k + \alpha = 0,$$

which is a particular integral of (7). Since the general integral of a Riccati equation can be derived when a particular integral is known, we are led to the non-oscillatory solution

$$f(x) = \frac{k}{x} + \frac{(2k+1)\,x^{2k}}{c - x^{2k+1}}$$

for $\alpha < \frac{1}{4}$, and the oscillatory solution

$$f(x) = \frac{1}{2x}\left\{2n\,\frac{c_2 \sin\log x^n - c_1 \cos\log x^n}{c_1 \sin\log x^n + c_2 \cos\log x^n} - 1\right\}$$

for $\alpha > \frac{1}{4}$, where $n = (\alpha - \frac{1}{4})^{1/2}$ and c, c_1, c_2 are arbitrary constants. For $\alpha = \frac{1}{4}$ we of course obtain the previously found general integral.

It may easily be verified that all the above solutions satisfy the appropriate functional equations, if the integration constants in the last two solutions are related to γ by

$$c^2 = \gamma^{2k+1} \qquad \text{and} \qquad \tan\log\gamma^n = \frac{2c_1c_2}{c_2{}^2 - c_1{}^2},$$

respectively.

Incidentally we notice that (9) has a solution of a much more general character, containing an infinite set of arbitrary constants; it is given by the formal Laurent series

$$f(x) = -\frac{1}{2x} + \sum_{n=0}^{\infty} d_n x^n \left\{1 - \left(\frac{\gamma}{x^2}\right)^{n+1}\right\},$$

where d_n are independent arbitrary constants.

The two features in the above example that seem remarkable are the inadvertently obtained value $\alpha = \frac{1}{4}$ which separates the two types of the solution curves, and the determination of the general integral of (1) for this value by merely differentiating the functional Eq. (8).

Appendix

Setting $a(x) \equiv z'(x)$, Eq. (5) becomes

$$\begin{aligned} z''b'' - 2z'bb'' + A_1b' + 3z'b'^2 + A_2b& \\ + A_3b^2 - 2z''b^3 + z'b^4 + A_4 = 0,& \end{aligned} \tag{A.1}$$

where

$$
\begin{aligned}
A_1 &= 4p(z)\, z'^3 - 4p(x)\, z' - z''', \\
A_2 &= 2p'(x)\, z' - 2p(x)\, z'' - 6p(z)\, z'^2 z'' - 2p'(z)\, z'^4, \\
A_3 &= 2p(x)\, z' + 2p(z)\, z'^3 + z''', \\
A_4 &= p(x)\{p(x)\, z' - 2p(z)\, z'^3 + z'''\} \\
&\qquad + p(z)\{p(z)\, z'^5 - z'^2 z''' + 3z' z''^2\} - p'(x)\, z'' + p'(z)\, z'^3 z'',
\end{aligned}
$$

and primes denote differentiation with respect to the indicated argument (either x or z).

Reference

1. M. Kuczma, "Functional Equations in a Single Variable," Polish Scientific Publishers, Warszawa, 1968.